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Separation of variables in path integrals and path integral solution of two potentials on the Poincaré upper half-plane

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Abstract. In this paper I discuss how to separate variables in path integrals. It is assumed that a one-dimensional problem with potential $V(x)$ has an exact solution with energy levels E_λ and wavefunctions Ψ_λ. In order to perform the separation of variables, a time transformation is performed back and forth in the path integral which allows one to insert the path integral solution corresponding to the potential $V(x)$. Furthermore, I illustrate the method by discussing some specific potential problems on the Poincaré upper half-plane. The first one is $V_1(x, y) = y^2[V(x) + (m/2)\omega^2 y^2]$, whereas the second is given by $V_2(x, y) = y^2(V(x) + \alpha/2my)$, which I call oscillator-like and Coulomb-like, respectively. $V(x)$ is an arbitrary one-dimensional potential. The various features of bound and continuous states are discussed. We find for V_1 that if bound states are to exist the energy spectrum of the one-dimensional problem, corresponding to $V(x)$, must have at least one negative energy level; for V_2 the existence of bound states is determined by the sign of α, i.e. $\alpha < 0$ is required.

1. Introduction

In this paper I develop a technique for separating variables in path integrals. For convenience I assume that a quantum mechanical potential problem $V(x)$ has an exact solution with energy levels E_λ and wavefunctions Ψ_λ. The level parameter λ may be discrete or continuous. By performing a time transformation in a path integral, the potential problem $V(x)$ is separated and its path integral solution can be inserted. Performing a second time transformation, which is in fact the inverse of the first, the energy E_λ then appears as an additional potential term in the remaining path integrations.

The motivation for developing such a technique emerges from the observation that many recent path integral calculations for multidimensional problems can be significantly simplified if a separation formula is available. Let us note, for example, the papers of Carpio-Bernido *et al* [1] for axial symmetric problems, Chetouani *et al* [2] for the Dyon, as well as the study of free motion on hyperbolic geometry [3, 4] (this short enumeration is far from being complete!).

In order to illustrate the separation technique I discuss the path integral formulations for two classes of potentials on the Poincaré upper half-plane U. The Poincaré upper half-plane U is defined as

$$U := \{(x, y) | y > 0, x \in \mathbb{R}\} \tag{1}$$

endowed with the hyperbolic line element ds

$$ds^2 = g_{ab}\, dq^a\, dq^b = \frac{dx^2 + dy^2}{y^2}. \tag{2}$$

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This model for a non-Euclidean geometry has recently become important in the theory of strings, where determinants of Laplacians on bounded domains on U arise in the multiloop perturbation expansion [5-8] and in the theory of quantum chaos in the connection with periodic orbit theory [9-12].

However, a study of the quantum mechanical properties on a curved space has its own legitimations, especially in a path integral formulation, where the motivation lies in the attempt to build up quantum mechanics 'from the point of view of fluctuating paths' [13]. There are already studies of potential problems on spaces of constant—positive and negative—curvature. Let us note the Kepler problem in a space of constant positive [14] and negative [15, 16] curvature. The free motion on U without [4, 12] and with magnetic field [17] has also been studied.

The first potential to be investigated is

$$V_1(x, y) = y^2\left(V(x) + \frac{m}{2}\omega^2 y^2\right) \tag{3}$$

which I call oscillator-like. The specific shape of $V(x)$ is left as general as possible. The second potential, which I call Coulomb-like, is

$$V_2(x, y) = y^2\left(V(x) + \frac{\alpha}{2my}\right). \tag{4}$$

We find that in order that bound states can exist, in the former potential $V(x) < 0$, whereas in the latter $\alpha < 0$ is required. Of course, these two potential problems are instructive in their own right in order to gain some insight into the features of hyperbolic geometry.

The further content of this paper will be as follows.

In the next section the technique of separation of variables in path integrals is developed. In the third section the oscillator-like potential will be discussed and in the fourth the Coulomb-like potential. The technique of separation of variables is, of course, applied. The corresponding path integrals are exactly evaluated and the wavefunctions and the energy spectra are explicitly stated. Whereas in the case of the oscillator-like potential my discussion will be more detailed, including a discussion of its connection to related problems, it is sufficient in the case of the Coulomb-like potential to proceed quite straightforwardly.

Section 5 contains a summary. In appendix 1 the orthonormality of the wavefunctions of the two potential problems is shown and in appendix 2 a dispersion relation involving Bessel functions is proven.

2. Formulation of the path integral and separation of variables

Before going into the analysis of the two potential problems and the separation of variables, let us sketch the formulation of path integrals on curved manifolds in general [18-27] and on the Poincaré upper half-plane in particular. Let us consider the generic case, where the classical Lagrangian is given by

$$L_{cl}(q, \dot{q}) = \tfrac{1}{2} m g_{ab}(q)\dot{q}^a\dot{q}^b - W(q) \tag{5}$$

with the metric tensor g_{ab}, the line element $ds^2 = q_{ab}dq^a dq^b$ and a potential $W(q)$. The quantum Hamiltonian has the form

$$H=-\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}}+W(q)=-\frac{\hbar^2}{2m}g^{-1/2}\partial_a g^{ab}g^{1/2}\partial_b+W(q) \tag{6}$$

where Δ_{LB} is the Laplace-Beltrami operator, $g=\det(g_{ab})$, $g^{ab}=(g_{ab})^{-1}$, and H, V and q denote D-dimensional coordinates. One considers momentum operators

$$p_a=\frac{\hbar}{i}\left(\frac{\partial}{\partial q^a}+\frac{\Gamma_a}{2}\right) \qquad \Gamma_a=\frac{\partial \log\sqrt{g}}{\partial q^a} \tag{7}$$

which are Hermitian with respect to the scalar product

$$(f_1,f_2)=\int\sqrt{g}\,\mathrm{d}q\,f_1^*(q)f_2(q). \tag{8}$$

The crucial point in the construction of the path integral is the ordering prescription one should use in the Hamiltonian. One well known ordering rule is the Weyl ordering, but I prefer an ordering prescription I called product ordering as used, for example, in [28] and more systematically developed in [29]. Here it is assumed that g_{ab} can be written as $g_{ab}=h_{ac}h_{cb}$, then

$$H=\frac{1}{2m}h^{ac}p_ap_bh^{cb}+W(q)+\Delta W(q) \tag{9}$$

with the well defined quantum potential

$$\Delta W=\frac{\hbar^2}{8m}[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_b)_{,a}+g^{ab}{}_{,ab}+2h^{ac}h^{bc}{}_{,ab}-h^{ac}{}_{,b}-h^{ac}{}_{,a}h^{bc}{}_{,b}h^{bc}{}_{,a}]. \tag{10}$$

The resulting D-dimensional path integral has the form

$$\begin{aligned}K(q'',q';T)=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon\hbar}\right)^{(N/2)D}\prod_{j=1}^{N-1}\int\sqrt{g(q^{(j)})}\,\mathrm{d}q^{(j)}\\ \times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\varepsilon}h_{ac}(q^{(j-1)})h_{cb}(q^{(j)})\Delta q^{(j)a}\Delta q^{(j)b}\right.\right.\\ \left.\left.-\varepsilon W(q^{(j)})-\varepsilon\Delta W(q^{(j)})\right)\right].\end{aligned} \tag{11}$$

Here $q^{(j)}=q(t'+\varepsilon j)$, $\varepsilon=(t''-t')/N=T/N$, $\Delta q^{(j)}=q^{(j)}-q^{(j-1)}$ in the limit $N\to\infty$. If not noted otherwise, every path integral in this paper must be interpreted in terms of (11). For $g_{ab}=f^2\delta_{ab}$, ΔW turns out to be quite simple

$$\Delta W=\hbar^2\frac{D-2}{8mf^4}\sum_a[(4-D)f_{,a}^2+2ff_{,aa}]. \tag{12}$$

For the Poincaré upper half-plane we obtain for the momentum operators

$$p_x=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} \qquad p_y=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial y}-\frac{1}{y}\right) \tag{13}$$

which are Hermitian with respect to the scalar product on U:

$$(f_1,f_2)=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y^2}f_1(x,y)f_2^*(x,y). \tag{14}$$

Functions $f\in L^2(U)$ must satisfy the boundary conditions $f\to 0$ for $y\to 0$. The Hamiltonian on U in the product ordering prescription is

$$H=-\frac{y^2\hbar^2}{2m}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)+V(x,y)=\frac{1}{2m}y(p_x^2+p_y^2)y+V(x,y) \tag{15}$$

with $\Delta W\equiv 0$ [cf equation (12)!]. $V(x, y)$ denotes some potential on the Poincaré upper half-plane, e.g. the potentials of (3) and (4). Thus we can write down the path integral on U in the 'product-form' definition, yielding

$$\begin{aligned}K(x'',x',y'',y';T)&=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon\hbar}\right)^N\prod_{j=1}^{N-1}\int_{-\infty}^{\infty}\mathrm{d}x^{(j)}\int_0^\infty\frac{\mathrm{d}y^{(j)}}{y^{(j)2}}\\&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^N\left(\frac{m}{2\varepsilon}\frac{\Delta^2x^{(j)}+\Delta^2y^{(j)}}{y^{(j-1)}y^{(j)}}-\varepsilon V(x^{(j)},y^{(j)})\right)\right].\end{aligned} \tag{16}$$

For solving path integrals several techniques must be applied. In particular I want to study how to separate variables in path integrals. Let us assume that the potential problem $V(x)$ has an exact solution according to

$$\int\mathscr{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)\right)\mathrm{d}t\right]=\int\mathrm{d}E_\lambda\exp(-\mathrm{i}E_\lambda T/\hbar)\Psi_\lambda^*(x')\Psi_\lambda(x''). \tag{17}$$

Here $\int\mathrm{d}E_\lambda$ denotes a Lebesque-Stieljes integral including discrete as well as continuous states. Now we consider the path integral

$$\begin{aligned}K(z'',z',x'',x';T)&=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int f^d(z^{(j)})\prod_{i=1}^{d'}g_i(z^{(j)})\,\mathrm{d}z_i^{(j)}\int\prod_{k=1}^{d}\mathrm{d}x_k^{(j)}\\&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^N\left[\frac{m}{2\varepsilon}\left(\sum_{i=1}^{d'}g_i(z^{(j-1)})g_i(z^{(j)})\Delta^2z_i^{(j)}+f(z^{(j-1)})f(z^{(j)})\sum_{k=1}^{d}\Delta^2x_k^{(j)}\right)\right.\right.\\&\quad\left.\left.-\varepsilon\left(\frac{V(x^{(j)})}{f^2(z^{(j)})}+W(z^{(j)})+\Delta W(z^{(j)})\right)\right]\right\}.\end{aligned} \tag{18}$$

Here $(z,x)\equiv(z_i,x_k)$ $(i=1,\ldots,d';\ k=1,\ldots,d,\ d'+d=D)$ denote a D-dimensional coordinate system, g_i and f the corresponding metric terms, and ΔW the quantum potential of (10). For simplicity I assume that the metric tensor g_{ab} involved has only diagonal elements, i.e. $g_{ab}=\mathrm{diag}[g_1^2(z),g_2^2(z),\ldots,g_{d'}^2(z),f^2(z),\ldots,f^2(z)]$. Of course, $\det(g_{ab})=f^{2d}\Pi_{i=1}^{d'}g_i^2\equiv f^{2d}G(z)$. The indices i and k will be omitted in the following. We perform the time transformation

$$s=\int_{t'}^{t}\frac{\mathrm{d}\sigma}{f^2[z(\sigma)]}\qquad s''=s(t''),\ s(t')=0 \tag{19}$$

where the lattice interpretation is $\varepsilon/[f(z^{(j-1)})f(z^{(j)})]=\delta^{(j)}\equiv\delta$. Of course, we identify $z(t)\equiv z[s(t)]$ and $x(t)\equiv x[s(t)]$. According to Kleinert [30] the transformation formulae for a pure time transformation are now given by

$$K(z'',z',x'',x';T)=\frac{1}{2\pi\mathrm{i}\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\mathrm{e}^{-\mathrm{i}ET/\hbar}G(z'',z',x'',x';E) \tag{20}$$

$$G(z'',z',x'',x';E)=\mathrm{i}[f(z')f(z'')]^{1-D/2}\int_0^\infty\mathrm{d}s''\,\tilde{K}(z'',z',x'',x';s'') \tag{21}$$

where the transformed path integral $\tilde{K}(s'')$ is given by

$$\begin{aligned}
\tilde{K}(z'', z', x'', x'; s'')
&= \int \frac{\sqrt{G(z)}}{f^{2-d}(z)} \mathcal{D}z(s) \int \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\left(\frac{g^2(z)}{f^2(z)}\dot{z}^2+\dot{x}^2\right) - V(x)\right.\right. \\
&\quad \left.\left. -f^2(z)(W(z)+\Delta W(z))+f^2(z)E\right]ds\right\} \\
&= \int dE_\lambda \exp(-iE_\lambda s''/\hbar)\Psi_\lambda^*(x')\Psi_\lambda(x'')\hat{K}(z'', z'; s'')
\end{aligned} \tag{22}$$

with the remaining path integration

$$\begin{aligned}
\hat{K}(z'', z'; s'') &= \int \frac{\sqrt{G(z)}}{f^{2-d}(z)} \mathcal{D}z(s) \\
&\quad \times \exp\left[\frac{i}{\hbar}\int_0^{s''}\left(\frac{m}{2}\frac{g^2(z)}{f^2(z)}\dot{z}^2 - f^2(z)(W(z)+\Delta W(z))+f^2(z)E\right)ds\right].
\end{aligned} \tag{23}$$

Of course, in the path integrals (22), (23) the same lattice formulation is assumed as in the path integral (18). Note the difference in comparison with a combined spacetime transformation [13, 21, 31-33] where a factor $[f(z')f(z'')]^{1/2}$ would appear instead. It is also seen that for $D=2$ the prefactor is identically 'one'. We perform a second time transformation in $\hat{K}(s'')$ effectively reversing the first:

$$\sigma = \int_0^s f^2[z(\omega)]\, d\omega \qquad \sigma'' = s'' \tag{24}$$

with the transformation on the lattice interpreted as $\sigma^{(j)} = \delta^{(j)} f(z^{(j-1)})f(z^{(j)})$. Therefore we obtain the transformation formulae

$$\hat{K}(z'', z'; s'') = \frac{1}{2\pi i\hbar}\int_{-\infty}^{\infty} dE' \exp(-iE's''/\hbar)\hat{G}(z'', z'; E') \tag{25}$$

$$\hat{G}(z'', z'; E') = i[f(z')f(z'')]^{(D-d)/2-1}\int_0^\infty d\sigma'' \exp(iE\sigma''/\hbar)\hat{\hat{K}}(z'', z'; \sigma'') \tag{26}$$

with the transformed path integral given by

$$\begin{aligned}
\hat{\hat{K}}(z'', z'; \sigma'') &= \int \sqrt{G(z)}\, \mathcal{D}z(\sigma) \\
&\quad \times \exp\left[\frac{i}{\hbar}\int_0^{\sigma''}\left(\frac{m}{2}g^2(z)\dot{z}^2 - W(z) - \Delta W(z) + \frac{E'}{f^2(z)}\right)d\sigma\right].
\end{aligned} \tag{27}$$

Plugging all the relevant formulae into (20) yields

$$\begin{aligned}
K(z'', z', x'', x'; T)
&= [f(z')f(z'')]^{-d/2}\int dE_\lambda\, \Psi_\lambda^*(x')\Psi_\lambda(x'') \\
&\quad \times \frac{1}{2\pi\hbar}\int_0^\infty d\sigma'' \int_{-\infty}^{\infty} dE \exp[-iE(\sigma''-T)/\hbar] \\
&\quad \times \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dE' \int_0^\infty ds'' \exp[-is''(E_\lambda+E')/\hbar] \int \sqrt{G(z)}\, \mathcal{D}z(\sigma) \\
&\quad \times \exp\left[\frac{i}{\hbar}\int_0^{\sigma''}\left(\frac{m}{2}g^2(z)\dot{z}^2 - W(z) - \Delta W(z) + \frac{E'}{f^2(z)}\right)d\sigma\right].
\end{aligned} \tag{28}$$

The $d\sigma'' \, dE$ integration produces just $\sigma'' \equiv T$, whereas the $dE' \, ds''$ integration can be evaluated by giving $E_\lambda + E$ a small negative imaginary part and applying the residuum theorem, yielding $E_\lambda \equiv -E'$ (for a similar discussion see, e.g., Chetouani and Hammann [34]). Therefore we arrive finally at the identity

$$K(z'', z', x'', x'; T) = [f(z')f(z'')]^{-d/2} \int dE_\lambda \, \Psi_\lambda^*(x')\Psi_\lambda(x'') \int \sqrt{G(z)} \, \mathcal{D}z(t) \times \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g^2(z)\dot{z}^2 - W(z) - \Delta W(z) - \frac{E_\lambda}{f^2(z)}\right)dt\right]. \tag{29}$$

Note that this result can be given a short-hand interpretation by inserting

$$\left(\frac{m}{2\pi i\delta^{(j)}\hbar}\right)^{d/2} \exp\left[\frac{i}{\hbar}\left(\frac{m}{2\delta^{(j)}}\Delta^2 x^{(j)} - \delta^{(j)}V(x^{(j)})\right)\right] = \int dE_{\lambda^{(j)}} \exp(-iE_{\lambda^{(j)}}\delta^{(j)}/\hbar)\Psi_{\lambda^{(j)}}^*(x^{(j-1)})\Psi_{\lambda^{(j)}}(x^{(j)}) \tag{30}$$

with $\delta^{(j)} = \varepsilon/[f(z^{(j-1)})f(z^{(j)})]$ for all j and applying the orthonormality of the Ψ_λ in each jth path integration. Equation (30) describes therefore a 'short-cut' to establishing (29) instead of performing a time transformation back and forth.

3. The oscillator-like potential

According to the introduction, the path integral formulation for the potential

$$V_1(x, y) = y^2\left(V(x) + \frac{m}{2}\omega^2 y^2\right) \tag{31}$$

on the Poincaré upper half-plane is given by

$$K(x'', x', y'', y'; T) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^N \prod_{j=1}^{N-1}\int_{-\infty}^{\infty} dx^{(j)} \int_0^\infty \frac{dy^{(j)}}{y^{(j)2}} \times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^N\left[\frac{m}{2\varepsilon}\frac{\Delta^2 x^{(j)} + \Delta^2 y^{(j)}}{\varepsilon y^{(j-1)}y^{(j)}} - \varepsilon y^{(j)2}\left(V(x^{(j)}) + \frac{m}{2}\omega^2 y^{(j)2}\right)\right]\right\}. \tag{32}$$

I apply (29) for the x-dependent potential in order to separate variables, and obtain

$$K(x'', x', y'', y'; T) = \int dE_\lambda \, \Psi_\lambda^*(x')\Psi_\lambda(x'')K_\lambda(y'', y'; T) \tag{33}$$

with the path integral $K_\lambda(T)$ given by

$$K_\lambda(y'', y'; T) = \sqrt{y'y}\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{N/2}\prod_{j=1}^{N-1}\int_0^\infty \frac{dy^{(j)}}{y^{(j)}} \times \exp\left[\frac{i}{\hbar}\sum_{j=1}^N\left(\frac{m}{2\varepsilon}\frac{\Delta^2 y^{(j)}}{y^{(j-1)}y^{(j)}} - \varepsilon\frac{m}{2}\omega^2 y^{(j)4} - \varepsilon E_\lambda y^{(j)2}\right)\right]. \tag{34}$$

This path integral can now be evaluated by two different methods:
(i) by a time transformation
(ii) by a coordinate transformation.

3.1. The time transformation

We perform the transformation

$$s(t)=\int_{t'}^{t} y^2(\sigma)\,\mathrm{d}\sigma \qquad s''=s(t''),\ s(t')=0 \tag{35}$$

so that $\varepsilon y^{(j-1)}y^{(j)}\equiv\delta^{(j)}\equiv\delta$. Furthermore, we identify $y(t)=y[t(s)]$. According to [30] we have the transformation formulae

$$K_\lambda(y'',y';T)=\frac{1}{2\pi\mathrm{i}\hbar}\int_{-\infty}^{\infty}\exp(-\mathrm{i}ET/\hbar)G_\lambda(y'',y';E)\,\mathrm{d}E \tag{36}$$

$$G_\lambda(y'',y';E)=\mathrm{i}\int_0^{\infty}\tilde{K}_\lambda(y'',y';s'')\exp(-\mathrm{i}s''E_\lambda/\hbar)\,\mathrm{d}s'' \tag{37}$$

where the transformed path integral $\tilde{K}_\lambda$ is given by (note that due to the time transformation the factor $\sqrt{y'y''}$ has been cancelled)

$$\begin{aligned}\tilde{K}_\lambda(y'',y';s'')&=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\delta\hbar}\right)^{N/2}\prod_{j=1}^{N-1}\int_0^{\infty}\mathrm{d}y^{(j)}\\&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\delta}\Delta^2y^{(j)}-\delta\frac{m}{2}\omega^2y^{(j)}+\delta\frac{E}{y^{(j)2}}\right)\right]\\&=\frac{m\omega\sqrt{y'y''}}{\mathrm{i}\hbar\sin\omega s''}\exp\left(-\frac{m\omega}{2\mathrm{i}\hbar}(y'^2+y''^2)\cot\omega s''\right)I_\nu\left(\frac{m\omega y'y''}{\mathrm{i}\hbar\sin\omega s''}\right)\end{aligned}\tag{38}$$

with $\delta=s''/N$ and $\nu=\sqrt{\frac14-2mE/\hbar^2}$. Here the well known path integral identity [35-37]

$$\begin{aligned}&\int\mathscr{D}r(t)\mu_\nu[r^2]\exp\left(\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2r^2)\,\mathrm{d}t\right)\\&\quad=\frac{m\omega\sqrt{r'r''}}{\mathrm{i}\hbar\sin\omega T}\exp\left(-\frac{m\omega}{2\mathrm{i}\hbar}(r'^2+r''^2)\cot\omega T\right)I_\nu\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)\end{aligned}\tag{39}$$

for radial path integrals has been applied with the functional measure $\mu_\nu[y^2]$

$$\begin{aligned}\mu_\nu[y^2]&=\lim_{N\to\infty}\prod_{j=1}^{N}\mu_\nu[y^{(j-1)}y^{(j)}]\\&=\lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{2\pi m y^{(j-1)}y^{(j)}}{\mathrm{i}\varepsilon\hbar}\right)^{1/2}\exp\left(-\frac{my^{(j-1)}y^{(j)}}{\mathrm{i}\varepsilon\hbar}\right)I_\nu\left(\frac{my^{(j-1)}y^{(j)}}{\mathrm{i}\varepsilon\hbar}\right)\end{aligned}\tag{40}$$

in order to guarantee a well defined short-time kernel [21, 37, 38]. I_ν describes a modified Bessel function. The path integral (34) has thus been transformed into the path integral of the radial harmonic oscillator. The sign of the square root must be chosen in such a way that

(i) for $E<\hbar^2/8m$: $\nu=+\sqrt{\frac14-2mE/\hbar^2}$ to guarantee the vanishing of the short-time kernel $\tilde{K}_\lambda$ for $y^{(j-1)}, y^{(j)}\to0$ in a powerlike behaviour;

(ii) for $E>\hbar^2/8m$: $\nu=-\mathrm{i}\sqrt{2mE/\hbar^2-\frac14}$ to guarantee that the Green function $G_\lambda(E)$ is the correct (retarded) one. Here one usually inserts $E\to E+\mathrm{i}\varepsilon$ ($\varepsilon>0$) whenever necessary, in order to deal with well defined formulae.

We now obtain for $G_\lambda(E)$:

$$G_\lambda(y'', y'; E) = \frac{m\omega\sqrt{y'y''}}{\hbar} \int_0^\infty \frac{ds''}{\sin \omega s''} I_\nu\left(\frac{m\omega y'y''}{\mathrm{i}\hbar \sin \omega\sigma''}\right) \times \exp\left(-\frac{m\omega}{2\mathrm{i}\hbar}(y'^2 + y''^2) \cot \omega s'' - \mathrm{i}s'' \frac{E_\lambda}{\hbar}\right). \tag{41}$$

Performing the transformation $u = \mathrm{i}\omega s''$, a Wick rotation and the second transformation $\sinh v = 1/\sinh u$, we obtain finally $(y'' \geqslant y')$

$$\begin{aligned} G_\lambda(y'', y'; E) &= \frac{m}{\hbar}\sqrt{y'y''} \int_0^\infty dv \left(\coth\frac{v}{2}\right)^{-E_\lambda/\hbar\omega} \\ &\quad \times \exp\left(-\frac{m\omega}{2\hbar}(y'2 + y''^2) \cosh v\right) I_\nu\left(\frac{m\omega}{\hbar} y'y'' \sinh v\right) \\ &= \frac{\Gamma[\frac{1}{2}(1+\nu+E_\lambda/\hbar\omega)]}{\sqrt{y'y''}\omega\Gamma(1+\nu)} W_{-E_\lambda/2\hbar\omega,\nu/2}\left(\frac{m\omega}{\hbar} y''^2\right) M_{-E_\lambda/2\hbar\omega,\nu/2}\left(\frac{m\omega}{\hbar} y'^2\right) \end{aligned} \tag{42}$$

where use has been made of the integral representation [39, p 729]

$$\begin{aligned} &\int_0^\infty \left(\coth\frac{v}{2}\right)^{2\nu} \exp\left(-\frac{a+b}{2} t \cosh v\right) I_{2\mu}(t\sqrt{ab} \sinh v)\, dv \\ &\qquad = \frac{\Gamma(\frac{1}{2}+\mu-\nu)}{t\sqrt{ab}\Gamma(1+2\mu)} W_{\nu,\mu}(at) M_{\nu,\mu}(bt). \end{aligned} \tag{43}$$

Here the $W_{\mu,\nu}(z)$ and $M_{\mu,\nu}(z)$ are Whittaker functions. Thus the complete Green function for the oscillator-like potential on the Poincaré upper half-plane is given by

$$\begin{aligned} &G(x'', x', y'', y'; E) \\ &\quad = \int dE_\lambda \Psi_\lambda^*(x')\Psi_\lambda(x'') \frac{\Gamma[\frac{1}{2}(1+\nu+E_\lambda/\hbar\omega)]}{\sqrt{y'y''}\omega\Gamma(1+\nu)} \\ &\qquad \times W_{-E_\lambda/2\hbar\omega,\nu/2}\left(\frac{m\omega}{\hbar} y'^2\right) M_{-E_\lambda/2\hbar\omega,\nu/2}\left(\frac{m\omega}{\hbar} y''^2\right). \end{aligned} \tag{44}$$

Poles occur in $G_\lambda(E)$ for

$$1 + \nu + \frac{E_\lambda}{\hbar\omega} = -2n \qquad n \in \mathbb{N}_0. \tag{45}$$

We have to distinguish between two cases:

(i) $E_\lambda > 0$: for positive ν the right-hand side is always positive and (45) allows no bound state solutions with $E < \hbar^2/8m$ at all;

(ii) $E_\lambda < 0$: bound states with $E < \hbar^2/8m$ exist and energy levels are given by

$$E_n = \frac{\hbar^2}{8m} - \frac{\hbar^2}{2m}\left(\frac{|E_\lambda|}{\hbar\omega} - 2n - 1\right)^2 \tag{46}$$

with $n = 0, 1, 2, \ldots, N_M < |E_\lambda|/2\hbar\omega - \frac{1}{2}$.

For the case (ii) the bound-state wavefunctions can be calculated with the help of the Hille-Hardy formula [39, p 1038]:

$$\frac{t^{-\lambda/2}}{1-t}\exp\left(-\frac{x+y}{2}\cdot\frac{1+t}{1-t}\right)I_\lambda\left(\frac{2\sqrt{xyt}}{1-t}\right) = \sum_{n=0}^{\infty}\frac{t^n n!}{\Gamma(n+\lambda+1)}(xy)^{\lambda/2}L_n^{(\lambda)}(x)L_n^{(\lambda)}(y)\exp[-\tfrac{1}{2}(x+y)] \tag{47}$$

where the $L_n(\lambda)$ are Laguerre polynomials. Inserting (47) into (41) and taking the residuum at each nth level yields the bound state contribution of the Green function as follows:

$$G^{(\text{bound})}(x'',x',y'',y';E)=\hbar\int \mathrm{d}E_\lambda\,\Psi_\lambda^*(x')\Psi_\lambda(x'')\sum_{n=0}^{N_M}\frac{\Psi_n(y')\Psi_n(y'')}{E_n-E} \tag{48}$$

with E_n as in (46) and the wavefunctions

$$\Psi_{\lambda,n}(x,y)=\left(\frac{2n!(|E_\lambda|/\hbar\omega-2n-1)y}{\Gamma(|E_\lambda|/\hbar\omega-n)}\right)^{1/2}\left(\frac{m\omega}{\hbar}y^2\right)^{|E_\lambda|/2\hbar\omega-n-1/2} \times\exp\left(-\frac{m\omega}{2\hbar}y^2\right)L_n^{(|E_\lambda|/\hbar\omega-2n-1)}\left(\frac{m\omega}{\hbar}y^2\right)\Psi_\lambda(x). \tag{49}$$

To calculate the continuous spectrum we insert into the Green function $G(E)$ the dispersion formula

$$\mathrm{i}\pi I_{-\mathrm{i}\sqrt{2mE/\hbar^2-1/4}}(z)=\int_{-\infty}^{\infty}\frac{pI_{-\mathrm{i}p}(z)\,\mathrm{d}p}{(p^2+\frac{1}{4})-2mE/\hbar^2} \tag{50}$$

instead of performing the s'' integration by means of (43). This relation is discussed in appendix 2. We obtain

$$\begin{aligned}
G^{(\text{cont})}&(x'',x',y'',y';E)\\
&=\frac{m\sqrt{y'y''}}{\mathrm{i}\hbar\pi}\int \mathrm{d}E_\lambda\,\Psi_\lambda^*(x')\Psi_\lambda(x'')\int_0^\infty \mathrm{d}v\left(\coth\frac{v}{2}\right)^{-E_\lambda/\hbar\omega}\\
&\quad\times\exp\left(-\frac{m\omega}{2\hbar}(y'^2+y''^2)\cosh v\right)\int_{-\infty}^{\infty}\frac{pI_{-\mathrm{i}p}((m\omega/\hbar)y'y''\sinh v)\,\mathrm{d}p}{(p^2+\frac{1}{4})-2mE/\hbar^2}\\
&=\hbar\frac{\sqrt{y'y''}}{\pi^2}\int \mathrm{d}E_\lambda\,\Psi_\lambda^*(x')\Psi_\lambda(x'')\int_0^\infty\frac{p\sinh\pi p\,\mathrm{d}p}{(\hbar^2/2m)(p^2+\frac{1}{4})-E}\\
&\quad\times\int_0^\infty \mathrm{d}v\left(\coth\frac{v}{2}\right)^{-E_\lambda/\hbar\omega}\exp\left(-\frac{m\omega}{2\hbar}(y'^2+y''^2)\cosh v\right)\\
&\quad\times K_{\mathrm{i}p}\left(\frac{m\omega}{\hbar}y'y''\sinh v\right)\\
&=\hbar\int \mathrm{d}E_\lambda\,\Psi_\lambda^*(x')\Psi_\lambda(x'')\int_0^\infty\frac{p\sinh\pi p\,\mathrm{d}p}{(\hbar^2/2m)(p^2+\frac{1}{4})-E}\\
&\quad\times\left|\Gamma\left[\frac{1}{2}\left(1+\mathrm{i}p+\frac{E_\lambda}{\hbar\omega}\right)\right]\right|^2\frac{\hbar}{2\pi^2 m\omega\sqrt{y'y''}}\\
&\quad\times W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{m\omega}{\hbar}y'^2\right)W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{mv}{\hbar}y''^2\right).
\end{aligned} \tag{51}$$

Here use has been made of the integral representation [39, p 729]

$$\int_0^\infty \left(\coth\frac{v}{2}\right)^{2\chi} \exp\left(-\frac{a+b}{2}t\cosh v\right) K_\mu(t\sqrt{ab}\sinh v)\,\mathrm{d}v$$
$$= \frac{\Gamma((1+\mu)/2-\chi)\Gamma((1-\mu)/2-\chi)}{2t\sqrt{ab}} W_{\chi,\mu/2}(at)\,W_{\chi,\mu/2}(bt). \tag{52}$$

The representation (51) shows clearly that $G(E)$ has a cut on the positive real axis in the complex energy plane with a branch point at $E = \hbar^2/8m$. From (51) we immediately read off the energy spectrum and the normalized wavefunctions

$$E_p = \frac{\hbar^2}{2m}\left(p^2+\frac{1}{4}\right) \tag{53}$$

$$\Psi_{p,\lambda}(x,y) = \left(\frac{\hbar}{m\omega}\frac{p\sinh\pi p}{2\pi^2 y}\right)^{1/2} \Gamma\left[\frac{1}{2}\left(1+\mathrm{i}p+\frac{E_\lambda}{\hbar\omega}\right)\right] W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{m\omega}{\hbar}y^2\right)\Psi_\lambda(x). \tag{54}$$

The orthonormality of the wavefunctions (49) and (54) can be shown by the two relations for the polynomials

$$\frac{n!\lambda}{\Gamma(n+\lambda+1)}\int_0^\infty \mathrm{e}^{-u}u^{\lambda-1}L_n^{(\lambda)}(u)L_m^{(\lambda)}(u)\,\mathrm{d}y$$
$$= \frac{n!\lambda}{\Gamma(n+\lambda+1)}\int_0^\infty \frac{\mathrm{d}u}{u^2} W_{n+(\lambda+1)/2,\lambda/2}(u)\,W_{m+(\lambda+1)/2,\lambda/2}(u) = \delta_{n,m} \tag{55}$$

and the scattering states

$$\frac{\sqrt{pp'\sinh\pi p\sinh\pi p'}}{4\pi^2}\Gamma[\tfrac{1}{2}(1+\mathrm{i}p+b)]\Gamma[\tfrac{1}{2}(1-\mathrm{i}p'+b)]$$
$$\times\int_0^\infty \frac{\mathrm{d}u}{u^2} W_{b,\mathrm{i}p/2}(u)\,W_{b,\mathrm{i}p'/2}(u) = \delta(p-p'). \tag{56}$$

These relations are discussed in appendix 1.

Therefore we obtain the path integral solution for an oscillator-like potential on the Poincaré upper half-plane

$$K(x'',x',y'',y';T)$$
$$= \int \mathrm{d}E_\lambda\left(\sum_{n=0}^{N_M}\exp(-\mathrm{i}E_nT/\hbar)\Psi^*_{\lambda,n}(x',y')\Psi_{\lambda,n}(x'',y'')\right.$$
$$\left.+\int_0^\infty \mathrm{d}p\,\exp(-\mathrm{i}TE_p/\hbar)\Psi^*_{\lambda,p}(x',y')\Psi_{\lambda,p}(x'',y'')\right) \tag{57}$$

with wavefunctions (49) and (54) and energy spectrum (46) and (53), respectively.

3.2. *The coordinate transformation*

For the second approach to calculating the path integral (34), I perform the coordinate transformations $q=\ln y$ and $2q\to\tilde q\equiv q$:

$$K_\lambda(q'',q';T)\equiv K_\lambda(e^{q''},e^{q'};T)$$
$$=\tfrac{1}{2}\exp\left(\frac{q'+q''}{4}-\frac{i\hbar T}{8m}\right)\lim_{N\to\infty}\left(\frac{M}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{j=1}^{N-1}\int_{-\infty}^{\infty}dq^{(j)}$$
$$\times\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\left(\frac{M}{2\delta}\Delta^2q^{(j)}-\delta\frac{M}{2}\Omega^2 e^{2q^{(j)}}-\delta E_\lambda e^{q^{(j)}}\right)\right].\tag{58}$$

Here $M=m/4$ and $\Omega=2\omega$. The additional factor $\exp(-i\hbar T/8m)$ is, of course, due to the nonlinear transformation and can be derived by the identity

$$\frac{im}{2\epsilon\hbar}\frac{\Delta^2y^{(j)}}{y^{(j-1)}y^{(j)}}=\frac{im}{2\hbar\epsilon}[\exp(\Delta q^{(j)}/2)-\exp(-\Delta q^{(j)}/2)]^2$$
$$\simeq\frac{im}{2\epsilon\hbar}\Delta^2q^{(j)}-\frac{m}{24i\epsilon\hbar}\Delta^4q^{(j)}$$
$$\doteq\frac{im}{2\epsilon\hbar}\Delta^2q^{(j)}-\frac{i\epsilon\hbar}{8m}\tag{59}$$

where use has been made of the identity $\Delta^4q^{(j)}\doteq 3(i\delta\hbar/m)^2$ (e.g. [18, 20, 25]) and I have used the symbol $\doteq$—following DeWitt [18]—to denote 'equivalence as far as use in the path integral is concerned'. This path integral is now the path integral for the Morse potential, which is actually calculated by the radial path integral identity (40). The path integral solution for the Morse potential (M) has the form [17, 28, 40, 41] $(k>0)$

$$\int\mathcal{D}q(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot q^2-\frac{\hbar^2k^2}{2m}\left(e^{2q}-2\frac{b}{\hbar k}e^q\right)\right]dt\right\}$$
$$=\sum_{n=0}^{N_M}\exp(-iTE_n^{(M)}/\hbar)\Psi_n^{(M)}(q')\Psi_n^{(M)}(x'')$$
$$+\int_0^\infty dp\exp(-iE_pT/\hbar)\Psi_p^{(M)*}(x')\Psi_p^{(M)}(x'')\tag{60}$$

where the bound-state wavefunctions and the energy spectrum are given by

$$\Psi_n^{(M)}(q)=\left(\frac{n!(2(b/\hbar)-2n-1)}{\Gamma(2(b/\hbar)-n)}\right)^{1/2}(2k\,e^q)^{b/\hbar-n-1/2}$$
$$\times\exp(-k\,e^q)L_n^{(2b/\hbar-2n-1)}(2k\,e^q)\tag{61}$$

$$E_n^{(M)}=-\frac{\hbar^2}{2m}\left(\frac{b}{\hbar}-n-\frac{1}{2}\right)^2\qquad n=0,1,\ldots,N_M<b/\hbar-\tfrac{1}{2}.\tag{62}$$

For the continuous spectrum one has

$$\Psi_p^{(M)}(q)=\left(\frac{p\sinh 2\pi p}{2\pi^2k}\right)^{1/2}\Gamma\left(ip-\frac{b}{\hbar}+\frac{1}{2}\right)e^{-q/2}W_{b,ip}(2k\,e^q)\tag{63}$$

with $E_p=\hbar^2p^2/2m$. Inserting this solution, performing the substitution $p\to p/2$ in the continuous spectrum, the results of (48), (51) are easily reproduced.

Let us make some remarks concerning the condition $E_\lambda<\hbar^2/8m$ in order that bound states can exist. We can look at this feature in two different ways.

(i) The bound state condition (45) quite clearly allows no bound states for $E_\lambda > \hbar^2/8m$. Calculating wavefunctions with the help of the Hille–Hardy formula nevertheless yields

$$\Psi^{(\lambda)}(x, y) = \left(\frac{2n!(E_\lambda/\hbar\omega + 2n+1)y}{\Gamma[n+1-\mathrm{i}(E_\lambda/\hbar\omega+2n+2)]}\right)^{1/2} \left(\frac{m\omega}{\hbar} y^2\right)^{-\mathrm{i}(E_\lambda/\hbar\omega+2n+1)} \times \exp\left(-\frac{m\omega}{\hbar} y^2\right) L_n^{[-\mathrm{i}(E_\lambda/\hbar\omega+2n+1)]}\left(\frac{m\omega}{\hbar} y^2\right) \tag{64}$$

which are not normalizable to unity with respect to the scalar product (14). This feature is well known and is due to the singularity of the potential $V = A/r^2$ in a path integral like (38). For $A < -\hbar^2/8m$ the potential is too strong and a particle moving in the field of such a potential with $E < 0$ 'falls into the centre' [42]. However, as Case [43] has pointed out, it is possible to construct 'quasi-bound' levels with energies $E_n \to -\infty$ corresponding to a particle which drops stepwise into the coordinate origin.

(ii) On the other hand, we see quite clearly the condition (45) arising from the path integral (58). For $E_\lambda > 0$ the Morse potential has no potential trough at all which could be able to produce bound states. Only for $E_\lambda < 0$ such a potential trough exists and thus bound states are allowed.

4. The Coulomb-like potential

In this section we study the quantum motion of a particle with mass m corresponding to the classical Lagrangian

$$L_{\mathrm{cl}}(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2} - y^2\left(V(x) + \frac{\alpha}{2my}\right) \tag{65}$$

where α is a positive or negative constant. To formulate the path integral on U, I use the product form

$$K(x'', x', y'', y'; T) = \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon\hbar}\right)^N \prod_{j=1}^{N-1}\int_{-\infty}^{\infty} \mathrm{d}x^{(j)} \int_0^\infty \frac{\mathrm{d}y^{(j)}}{y^{(j)2}} \times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^N \left[\frac{m}{2}\frac{\Delta^2 x^{(j)} + \Delta^2 y^{(j)}}{\varepsilon y^{(j-1)} y^{(j)}} - \varepsilon y^{(j)2}\left(V(x^{(j)}) + \frac{\alpha}{2my^{(j)}}\right)\right]\right\}. \tag{66}$$

I proceed similarly to the previous section and separate variables according to (29), yielding

$$K(x'', x', y'', y'; T) = \int \mathrm{d}E_\lambda\, \Psi_\lambda^*(x')\Psi_\lambda(x'') K_\lambda(y'', y'; T) \tag{67}$$

with the path integral $K_\lambda(T)$ given by

$$K_\lambda(y'', y'; T) = \sqrt{y'y''}\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon\hbar}\right)^{N/2} \prod_{j=1}^{N-1}\int_0^\infty \frac{\mathrm{d}y^{(j)}}{y^{(j)}} \times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^N \left(\frac{m}{2\varepsilon}\frac{\Delta^2 y^{(j)}}{y^{(j-1)}y^{(j)}} - \varepsilon\frac{\alpha}{2m} y^{(j)} - \varepsilon E_\lambda y^{(j)2}\right)\right]. \tag{68}$$

Again we can calculate this path integral by two different methods.

(i) The same time transformation as in section 2, yielding effectively the path integral for the Coulomb potential.

(ii) The coordinate transformation $q = \ln y$.

In the first case we obtain, similar to (38),

$$\tilde{K}_\lambda(y'', y'; s'') = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta\hbar}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^\infty dy^{(j)} \times \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\delta}\Delta^2 y^{(j)} - \delta\frac{\alpha}{2my^{(j)}} + \delta\frac{E}{y^{(j)2}}\right)\right] \tag{69}$$

and the path integral solutions for the Coulomb potential [13, 33, 44-47] can be applied. As for the Morse potential, the path integral calculation for the Coulomb potential is based on the identity (39). This justifies the notion 'Coulomb-like'. We obtain for the second method again the path integral for the Morse potential with $V(q) = (\hbar^2k^2/2m)[e^{2q} + \alpha\, e^q/(\hbar k)^2]$ and $\hbar^2k^2/2m = E_\lambda$:

$$\hat{K}_\lambda(q'', q'; T) = K_\lambda(e^{q''}, e^{q'}; T) = \exp\left(\frac{q'+q''}{2} - \frac{i\hbar T}{8m}\right)\int \mathcal{D}q(t) \times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2 - \frac{\hbar^2k^2}{2m}\left(e^{2q} + \frac{\alpha}{(\hbar k)^2}e^q\right)\right]dt\right\}. \tag{70}$$

Using the solution for the Morse potential yields

$$K(x'', x', y'', y'; T) = \int dE_\lambda\left(\sum_{n=0}^{N_M}\exp(-iTE_n/\hbar)\Psi^*_{\lambda,n}(x', y')\Psi_{\lambda,n}(x'', y'') + \int_0^\infty dp\,\exp(-iTE_p/\hbar)\Psi^*_{\lambda,p}(x', y')\Psi_{\lambda,p}(x'', y'')\right). \tag{71}$$

The wavefunctions and energy spectrum are given for the continuous states

$$\Psi_{\lambda,p}(x, y) = \left(\frac{p\sinh 2\pi p}{2\pi^2 k}\right)^{1/2}\Gamma\left(\frac{1}{2} + ip + \frac{\alpha}{2(\hbar k)^2}\right)W_{-\alpha/2(\hbar k)^2, ip}(2ky)\Psi_\lambda(x) \tag{72}$$

$$E_p = \frac{\hbar^2}{2m}(p^2 + \tfrac{1}{4}). \tag{73}$$

For the bound states we obtain similarly ($n = 0, 1, \ldots, N_M < \frac{1}{2}(|\alpha|/(\hbar k)^2 - 1)$)

$$\Psi_{\lambda,n}(x, y) = \sqrt{\frac{(|\alpha|/(\hbar k)^2 - 2n - 1)n!}{4\pi|k|\Gamma(|\alpha/(\hbar k)^2| - n)}} \times e^{-ky}(2ky)^{|\alpha|/(\hbar k)^2 - n}L_n^{(|\alpha|/(\hbar k)^2 - 2n - 1)}(2ky)\Psi_\lambda(x) \tag{74}$$

$$E_n = \frac{\hbar^2}{8m} - \frac{\hbar^2}{2m}\left(\frac{|\alpha|}{2(\hbar k)^2} - n - \frac{1}{2}\right)^2. \tag{75}$$

Bound states can only exist if $\alpha < 0$ and $E_\lambda > 0$. Of course, both alternatives (i) and (ii) lead to the same result and it remains a matter of taste which one is actually chosen. The connection between the path integrals (34) and (69) shows once again the close connection between the solutions of the Morse potential, Coulomb potential and the radial harmonic oscillator, which is, of course, due to the fact that the corresponding

Schrödinger equations can be reduced to the differential equation for the confluent hypergeometric function.

5. Summary

In this paper I have discussed how to separate variables in path integrals. We perform in a given path integral a time transformation in such a way that a separated path integral emerges so that we could insert its solution. Performing a second time transformation which was the reverse of the former, the original problem was reduced by the separated variables. However, the energy of the separated problem now occurs as a potential term in the remaining path integrations.

In addition I have illustrated the method for specific potential problems on the Poincaré upper half-plane, which I have called oscillator-like and Coulomb-like. Equivalently, both problems could also be formulated in terms of the Morse potential path integral. For the oscillator-like potential we found a sensitive dependence on the corresponding one-dimensional potential problem $V(x)$, which has been included and left open in its specific shape in order to be as general as possible. Bound state levels can only exist if the potential $V(x)$ has negative energy levels. As is known, the Hamiltonian on U with a positive potential $V(x, y)$ has as lower bound

$$H \geqslant H|_{V=0} > \frac{\hbar^2}{8m} \equiv H_0. \qquad (76)$$

This can be seen if one considers, for example, the classical Hamiltonian corresponding to the Lagrangian L and inserts the Heisenberg uncertainty relations $xp_x \geqslant \hbar/2$ and $yp_y \geqslant \hbar/2$. Thus a negative term is needed to lower the energy of the Hamiltonian below the critical value H_0.

The same job of changing the lower bound of H is done in the case of the Coulomb-like potential by the Coulomb term itself, so that $\alpha < 0$ is required. Here in turn, $E_\lambda > 0$, so that just the opposite of the previous case is needed (otherwise $k^2 < 0$ and therefore k is purely imaginary). Of course, all the results can also be achieved by solving the corresponding Schrödinger equations, as is easily checked.

Most clearly these features appear after the coordinate transformations in the two cases, where both problems are transformed into Morse potentials and the effective shape of the trough of the Morse potential determines whether there are bound states or not.

The two potential problems on the Poincaré upper half-plane have been studied on the one hand to illustrate the method of separating variables in path integrals, and on the other to gain some insight into the features of hyperbolic geometry. Of course, they are simple examples. The power of the separation technique can be even better illustrated by looking at some recent path integrations by, e.g., Carpio-Bernido *et al* [1], where potentials with axial symmetry are considered, as well as Chetouani *et al* [2] in a path integral treatment for the Dyon. These authors perform quite reasonable and formidable transformations to tackle these problems and actually separate variables (e.g. the angular from the radial path integration). Here the separation formula (29) comes into play, making these problems in its very evaluation quite easy.

Another less simple example for the application of the separation formula (29) is the path integration on the $SU(n-v, v)$ $(n > v, n \geqslant 3)$ group manifold. Formulated in an appropriate polar coordinate system, this problem gives rise to at first sight rather involved angular path integrations, which, however, can be successively performed by applying formula (29) in a straightforward way [48].

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Appendix 1. Discussion of the Orthonormality of the Wavefunctions

I only consider the wavefunctions for the continuous spectrum of the oscillator-like potential explicitly. The orthonormality of the corresponding bound-state and continuous wavefunctions of the Coulomb-like potential is treated similarly. Inserting the wavefunctions (54) into the scalar product (14) yields

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \frac{\mathrm{d}y}{y^2} \Psi_{\lambda,p}(x,y)\Psi^*_{\lambda',p'}(x,y) &= \delta_{\lambda,\lambda'} \frac{\Gamma((1+\mathrm{i}p-b)/2)\Gamma((1-\mathrm{i}p'-b)/2)}{4\pi^2}\sqrt{pp'\sinh\pi p\sinh\pi p'} \\ &\quad\times \int_0^{\infty} \frac{du}{u^2} W_{b,\mathrm{i}p/2}(u)\, W_{b,\mathrm{i}p'/2}(u) \end{aligned} \tag{A1.1}$$

where I have changed variables $(m\omega/\hbar)y^2 \to u$ and abbreviated $b = -E_\lambda/2\hbar\omega$. $\delta_{\lambda,\lambda'}$ is shorthand for the orthonormality of the functions Ψ_λ, whether they are discrete or continuous wavefunctions. The remaining integral can be evaluated with the help of [39, p 858]:

$$\begin{aligned}\int_0^{\infty} x^{\rho-1} W_{\kappa,\mu}(x)\, W_{\lambda,\nu}(x)\,\mathrm{d}x &= \frac{\Gamma(1+\mu+\nu+\rho)\Gamma(1-\mu+\nu+\rho)\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\lambda-\nu)\Gamma(\frac{3}{2}-\kappa+\nu+\rho)} \\ &\quad\times {}_3F_2(1+\mu+\nu+\rho, 1-\mu+\nu+\rho, \tfrac{1}{2}-\lambda+\nu; 1+2\nu, \tfrac{3}{2}-\kappa+\nu+\rho; 1) \\ &\quad+\frac{\Gamma(1+\mu-\nu+\rho)\Gamma(1-\mu-\nu+\rho)\Gamma(2\nu)}{\Gamma(\frac{1}{2}-\lambda+\nu)\Gamma(\frac{3}{2}-\kappa-\nu+\rho)} \\ &\quad\times {}_3F_2(1+\mu-\nu+\rho, 1-\mu-\nu+\rho, \tfrac{1}{2}-\lambda-\nu; 1-2\nu, \tfrac{3}{2}-\kappa-\nu+\rho; 1). \end{aligned} \tag{A1.2}$$

Setting $\rho = \varepsilon - 1$ ($\varepsilon \ll 1$), $\kappa = \lambda = b$, $\mu = \mathrm{i}p$ and $\nu = \mathrm{i}p'$, I obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^{\infty} \frac{\mathrm{d}u}{u^{2-\varepsilon}} W_{b,\mathrm{i}p/2}(u)\, W_{b,\mathrm{i}p'/2}(u) &= \frac{\Gamma(\mathrm{i}p)\Gamma(-\mathrm{i}p')}{\Gamma((1+\mathrm{i}p-b)/2)\Gamma((1-\mathrm{i}p'-b)/2)} \\ &\quad\times \lim_{\varepsilon\to 0}\{\Gamma[\varepsilon+(\mathrm{i}/2)(p-p')]+\Gamma[\varepsilon-(\mathrm{i}/2)(p-p')]\} \\ &= 4\pi\left|\frac{\Gamma(\mathrm{i}p)}{\Gamma((1-b+\mathrm{i}p)/2)}\right|^2 \delta(p-p'). \end{aligned} \tag{A1.3}$$

In the above calculation I have used the fact that in the limit $\varepsilon \to 0$ the function ${}_3F_2$ changes into ${}_2F_1$, which can be evaluated at $z=1$ with the help of [39, p 1042]

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \varepsilon \to 0 \tag{A1.4} $$

for $c = 1 + \mathrm{i}p'$, $a = \varepsilon + (\mathrm{i}/2)(p+p')$ and $b = \varepsilon + (\mathrm{i}/2)(p'-p)$. Combining (A1.3) and (A1.4) shows finally the orthonormality relation (56). Due to the relation (55) the orthonormality of the wavefunctions (49) is shown similarly.

Appendix 2. Proof of the dispersion relation (50)

We consider the complex contour integral (let $E, \lambda > 0$)

$$ \oint_C \frac{zI_{-\mathrm{i}z}(\lambda)}{z^2+\frac{1}{4}-2mE/\hbar^2}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{zI_{-\mathrm{i}z}(\lambda)}{z^2+\frac{1}{4}-2mE/\hbar^2}\right) \tag{A2.1} $$

where its value is given by the residuum theorem. A slightly different discussion of this integral was already given in [17]. For the poles in the complex plane we choose the convention $E \to E + \mathrm{i}\varepsilon$, $(0 < \varepsilon \ll 1$ so that the poles of the integrand of the integral (A2.1) are located at $z_1 = \sqrt{2mE/\hbar^2 - \frac{1}{4}} + \mathrm{i}\delta$, $z_1 = -\sqrt{2mE/\hbar^2-\frac{1}{4}} - \mathrm{i}\delta$ $(0 < \delta = \delta(\varepsilon) \ll 1)$. I take for C the closed contour

$$ C: \begin{cases} z = p & p \in [-R, R] \\ z = R\,\mathrm{e}^{\mathrm{i}\phi} & \phi \in (0, \pi) \end{cases} \tag{A2.2} $$

and consider the limit $R \to \infty$. If it can be shown that the integral over the semicircle vanishes; I obtain

$$ \int_{-\infty}^{\infty} \frac{pI_{-\mathrm{i}p}(\lambda)\,\mathrm{d}p}{p^2+\frac{1}{4}-2mE/\hbar^2} = \mathrm{i}\pi I_{-\mathrm{i}\sqrt{2mE/\hbar^2-1/4}}(\lambda) \tag{A2.3} $$

which is the integral we need. For the integral over the semicircle we obtain for R finite

$$ \begin{aligned} |I_{\text{semicircle}}| &= \left| \mathrm{i}R^2 \int_0^{\pi} \frac{\mathrm{e}^{2\mathrm{i}\phi} I_{-\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}}(\lambda)\,\mathrm{d}\phi}{R^2\mathrm{e}^{2\mathrm{i}\phi}+\frac{1}{4}-2mE/\hbar^2} \right| \\ &\leqslant 2\pi \max_{\phi\in(0,\pi)} |I_{-\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}}(\lambda)|. \end{aligned} \tag{A2.4} $$

With the asymptotic expansion of the modified Bessel function [49, p 122] for high order

$$ I_\nu(\nu z) \simeq \frac{(1+z^2)^{-1/4}}{\sqrt{2\pi\nu}} \exp\left(\nu\sqrt{1+z^2} + \nu \ln \frac{z}{1+\sqrt{1+z^2}}\right) \qquad \nu \to \infty \tag{A2.5} $$

we see that the main contribution comes from the factor $\mathrm{e}^{\nu \ln z}$. Inserting the relevant terms, I get

$$ \begin{aligned} |I_{\text{semicircle}}| &\leqslant \left(\frac{\pi}{R}\right)^{1/2} \\ &\quad \times \exp[-R\sin\phi(\ln R - \ln\lambda) + 10R] \to 0 \qquad \phi \in (0,\pi),\ R \to \infty. \end{aligned} \tag{A2.6} $$

Thus the integral (A2.4) vanishes in the limit $R \to \infty$ and therefore (A2.3) is proven. Note that if the + sign is used in $I_{+\mathrm{i}z}$ one has to replace $E \to E - \mathrm{i}\varepsilon$ $(0 < \varepsilon \ll 1)$, which leads to an advanced Green function.

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